Pressure Behavior of a Limited Circular Composite Reservoir

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ABSTRACT

The analytical solution is presented to the problem of flow of a slightly compressible fluid in a limited, composite reservoir with radial symmetry which is produced by a well at the center. Numerical results are given for a specific case. It is believed that this type of heterogeneity can account for some actually observed pressure behavior and should be of special value in the interpretation of reservoir limit tests.

The system of interest is composed of two zones of different permeability in concentric series. There is no flow across the outer boundary of the outer zone, and fluid is withdrawn from the system at a well represented by a point sink located at the center of the inner zone. The solution to this problem is useful in the fundamental study of the behavior of reservoirs having a low permeability "rim" and in the pressure transient behavior of some wells having a large horizontal fracture or a large fractured area such as would be created by a nuclear explosion. The analytical solution to this particular problem has apparently not been previously published.

This paper contains (1) a mathematical statement of the problem, (2) the analytical solution, (3) numerical results for a specific problem, and (4) discussion of the physical interpretation of these results. The Appendix contains descriptions of the procedures used to obtain the analytical solution and the tabulated results for a specific problem. The specific numerical results given show that reservoir fluid from a very low permeability rim can contribute to production from a well located in the high permeability area. Predicted pressure drawdown and build-up behavior for the system is given.

INTRODUCTION

The present work was undertaken to develop a basis for interpretation of some observed well pressure transient behavior that did not appear to be otherwise explainable. The solution described herein has special significance in the interpretation of well pressure transient tests designed to indicate reservoir limits.

The partial differential equations describing transient heat conduction and the transient flow of fluids having small and constant compressibility in porous media are mathematically identical. Analytical solutions to this equation for systems involving media of different conductivities in concentric series have appeared in connection with both types of problems. In the heat conduction literature, solutions have been published by Jaeger[1] and Carslaw and Jaeger.[2] In reservoir engineering literature, solutions have been published by Hazebroek, Matthews and Rainbow,[3] Hurst[4] and Loucks and Guerrero.[5] Additional published solutions are cited in Ref. 2. In all of these references the Laplace transform method equivalent to that introduced to the petroleum literature by van Everdingen and Hurst[10] has been utilized to obtain solutions. This method was also employed in the present work.

Although radial symmetry is specifically assumed in this treatment, the numerical results should give some qualitative insight into the behavior of many reservoirs having a roughly circular area of commercial pay surrounded by a hydrocarbon-containing region in which wells would be non-commercial because of low permeability.

Hopkinson *et al.*[8] gave an expression for the linear asymptote portion of the solution for the linear zone which is equivalent to the one given in this paper. Ref. 8 considers a ratio of diffusivities between the two zones which may be independent of the ratio of permeabilities. In this paper, a difference in permeabilities only is considered and the ratio of the hydraulic diffusivities in the two zones is equal to the ratio of the permeabilities.

STATEMENT OF THE PROBLEM

The problem for which a solution is wanted is defined by the following equations. A diagram of the reservoir system is shown in Fig. 1:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Delta p_1}{\partial r}\right)=\frac{\partial \Delta p_1}{\partial t_D}\quad,(0<r\leq r_1)\qquad \text{(1a)}$$

[1]References given at end of paper.

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$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Delta p_2}{\partial r}\right) = F_k \frac{\partial \Delta p_2}{\partial t_D}\ , \quad (r_1 \leq r \leq 1) \qquad \text{(1b)}$$

$$\lim_{r \to 0}\left(r\frac{\partial \Delta p_1}{\partial r}\right) = -1 \qquad \text{(2a)}$$

$$\frac{\partial \Delta p_1(r_1, t_D)}{\partial r} = \frac{1}{F_k}\frac{\partial \Delta p_2(r_1, t_D)}{\partial r} \qquad \text{(2b)}$$

$$\Delta p_1(r_1, t_D) = \Delta p_2(r_1, t_D) \qquad \text{(2c)}$$

$$\frac{\partial \Delta p_2(1, t_D)}{\partial r} = 0 \qquad \text{(2d)}$$

$$\Delta p_1(r, 0) = \Delta p_2(r, 0) = 0 \qquad \text{(3)}$$

STATEMENT OF THE SOLUTION

The solution is:

$$\Delta p_1(r, t_D) = 2t_D + \ln\frac{r_1}{r} + \frac{r^2}{2} - F_k\left[\ln r_1 + \frac{3}{4}\right] + (F_k - 1)\left[r_1^2 - \frac{r_1^4}{4}\right] + \sum_{j=1}^{\infty} B_j e^{-\alpha_j^2 t_D} \qquad \text{(4a)}$$

$$\Delta p_2(r, t_D) = 2t_D + F_k\left[\frac{r^2}{2} - \ln r - \frac{3}{4}\right] + (F_k - 1)\left[\frac{r_1^2}{2} - \frac{r_1^4}{4}\right] + \sum_{j=1}^{\infty} C_j e^{-\alpha_j^2 t_D} \qquad \text{(4b)}$$

$$B_j = \left(\frac{2}{\alpha_j^2}\right)\left[\frac{J_0(r\alpha_j)}{r_1^2(F_k-1)J_1^2(r_1\alpha_j) - \left(\frac{4}{\Pi^2\alpha_j^2 F_k}\right)\left(\frac{J_0^2(r_1\alpha_j)}{\Phi_{01}^2}\right)}\right] \qquad \text{(4c)}$$

$$C_j = \left(\frac{2}{\alpha_j^2}\right)\left[\frac{Y_0(r\alpha_j\sqrt{F_k})J_1(\alpha_j\sqrt{F_k}) - J_0(r\alpha_j\sqrt{F_k})Y_1(\alpha_j\sqrt{F_k})}{\frac{J_0(r_1\alpha_j)}{\Phi_{01}}\left(r_1^2(F_k-1)\Phi_{01}^2\frac{J_1^2(r_1\alpha_j)}{J_0^2(r_1\alpha_j)} - \frac{4}{\Pi^2\alpha_j^2 F_k}\right)}\right] \qquad \text{(4d)}$$

where α_j is the root of

$$J_0(r_1\alpha_j)\Phi_{11} - \sqrt{F_k}\ J_1(r_1\alpha_j)\Phi_{01} = 0 \qquad \text{(4e)}$$

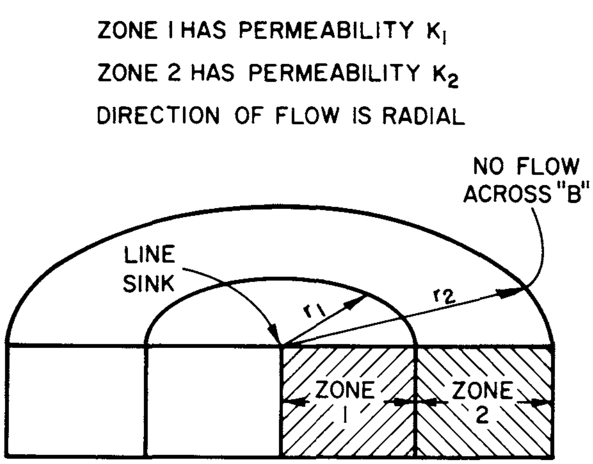

FIG. 1 — DIAGRAM OF COMPOSITE SYSTEM.

NUMERICAL RESULTS FOR A SPECIFIC CASE

Problem conditions for a specific case are listed in Table 1 which also contains the numerical results. Table 2 is a list of the first 48 values of α_j for this case. The dimensionless solution presented and other dimensionless solutions which

TABLE 1 — CALCULATED VALUES OF $\Delta p\ (r, t_D)$ FOR $r_1 = 0.5$, $F_k = 200$

t_D	$r = 0.001$	$r = 0.1$	$r = 0.3$	$r = 0.5$	$r = 0.7$	$r = 0.9$	$r = 1.0$
0	0	0	0	0	0	0	0
0.001	*	*	*	—	—	—	—
0.0025	*	0.112	*	—	—	—	—
0.005	*	0.283	0.006	—	—	—	—
0.01	5.012	0.522	0.017	—	—	—	—
0.025	5.467	0.911	0.130	0.024	—	—	—
0.05	5.819	1.240	0.334	0.154	—	—	—
0.1	6.248	1.662	0.716	0.504	—	—	—
0.25	7.336	2.748	1.789	1.556	—	—	—
0.5	9.021	4.432	3.463	3.212	0.001	—	—
1.0	12.123	7.533	6.553	6.279	0.060	—	—
2.5	20.232	15.640	14.641	14.329	1.065	0.026	0.005
5.0	31.524	26.930	25.914	25.568	4.489	0.499	0.250
10.0	49.664	45.067	44.033	43.651	13.328	3.867	2.963
25.0	88.975	84.375	83.322	82.901	42.355	25.973	24.151
50.0	141.295	136.694	135.636	135.206	92.120	73.999	71.946
100.0	241.484	236.883	235.825	235.394	192.101	173.839	171.767

$^*\Delta p\ (r, t_D) = \frac{1}{2}\left[-E_i\left(-\frac{r^2}{4t_D}\right)\right]$

TABLE 2 — VALUES OF α_j FOR THE PROBLEM OF TABLE 1

j	α_j	j	α_j	j	α_j
1	0.3213	17	7.2816	33	13.8862
2	0.7125	18	7.6070	34	14.1307
3	1.1386	19	7.8626	35	14.4819
4	1.5744	20	8.2522	36	14.9052
5	2.0134	21	8.6819	37	15.3412
6	2.4541	22	9.1196	38	15.7809
7	2.8956	23	9.5599	39	16.2220
8	3.3375	24	10.0013	40	16.6637
9	3.7797	25	10.4432	41	17.1057
10	4.2221	26	10.8853	42	17.5478
11	4.6644	27	11.3273	43	17.9897
12	5.1066	28	11.7692	44	18.4311
13	5.5483	29	12.2104	45	18.8714
14	5.9893	30	12.6504	46	19.3093
15	6.4286	31	13.0873	47	19.7396
16	6.8635	32	13.5141	48	20.1330

may be obtained to the problem are related to the behavior of actual physical systems by the following equations:

$$\Delta p_{1a}(r,t) = \left[\frac{q\mu B_o}{2\pi k_1 h}\right] \Delta p_1(r,t_D) \quad . \text{(5a)}$$

$$\Delta p_{2a}(r,t) = \left[\frac{q\mu B_o}{2\pi k_1 h}\right] \Delta p_2(r,t_D) . \text{(5b)}$$

$$t = \left(\frac{\Phi\mu c r_{2a}^2}{k_1}\right) t_D \quad \ldots\ldots\ldots \text{(5c)}$$

To illustrate the use of these equations, consider a system with the following properties and conditions:

initial pressure = 4,000 psia
wellbore radius = 1 ft
inner zone radius = 500 ft
outer zone radius = 1,000 ft
$c = 1.44 \times 10^{-4}$ atm^{-1}
$\mu = 1$ cp
$B_o = 1.21$
flow rate = 8.27 B/D
thickness = 10 ft
$k_1 = 0.01$ darcies
$k_2 = 5 \times 10^{-5}$ darcies
porosity = 0.1

This system has the same values of r_1 and F_k as the one for which a dimensionless solution was obtained. Hence the results given in Table 1 can be used.

Figs. 2 and 3 contain the wellbore drawdown history. Fig. 4 shows pressure distribution at 38.7 and 387 days. Fig. 5 shows the build-up history if the well is shut in after 154.8 days of

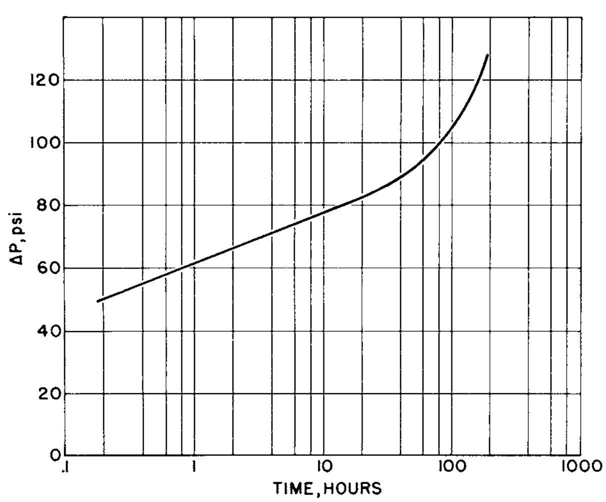

FIG. 2 — BOTTOM-HOLE PRESSURE DROP FOR EXAMPLE CASE.

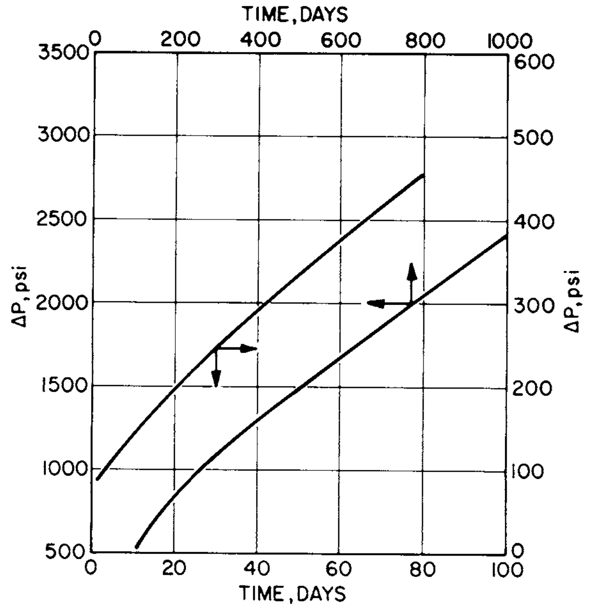

FIG. 3 — BOTTOM-HOLE PRESSURE DROP FOR EXAMPLE CASE.

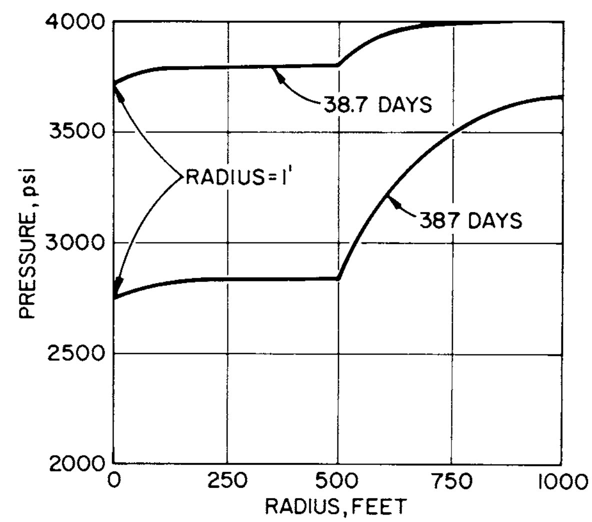

FIG. 4 — PRESSURE DISTRIBUTION FOR EXAMPLE CASE.

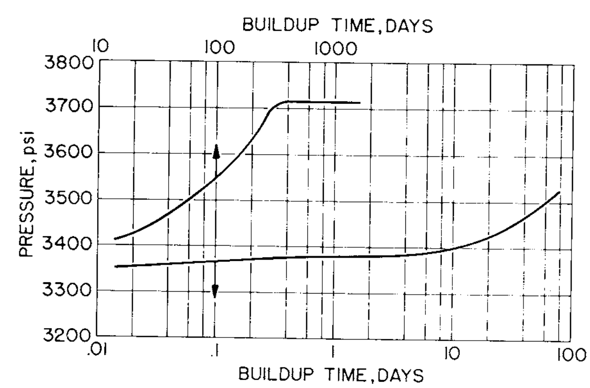

FIG. 5 — BUILD-UP HISTORY FOR EXAMPLE CASE (SHUT IN AFTER 154.8 DAYS OF PRODUCTION).

production. The build-up history is obtained by using the drawdown curve and the principle of superposition. Fig. 6 is a plot of the rate of change of bottom-hole flowing pressure with cumulative production as a function of cumulative production.

INTERPRETATION OF RESULTS

Fig. 2 is a conventional semi-log plot of the early portion of the constant rate drawdown history for the example problem. The straight line portion, which lasts for about 20 hours, can be used to estimate flow capacity in the inner zone, but the ratio of outer zone radius (r_{2a}) to inner zone radius (r_{1a}) is not large enough for a straight line segment to be formed in the drawdown curve from which a reliable kh value for the outer zone could be calculated. In systems in which the ratio (r_{2a}/r_{1a}) is sufficiently large, Hurst[4] has shown that kh in the outer zone can be determined in this way.

Fig. 3 is a plot of drawdown as a function of time on a rectilinear graph. In the period from 40 to 80 days, a nearly linear relationship has apparently developed which would indicate that the reservoir limits had been reached and an entire reservoir containing about 0.208×10^6 STB was undergoing uniform depletion. However, the later drawdown history beginning at about 500 days is a straight line with a flatter slope indicating uniform depletion of a reservoir containing about 0.463×10^6 STB. The first apparently straight line indicates uniform depletion of a volume somewhat greater than that contained in the inner zone while the final straight line indicates uniform depletion of a volume equal to that of the entire reservoir (0.463×10^6 STB).

Fig. 4 shows pressure distribution at 38.7 and 387 days. The shape of these figures is that which would be intuitively anticipated. An unexpected result is that uniform depletion does not completely occur until considerable time has elapsed after the transient has reached the outer boundary. The explanation is that the rate at which fluid flows from the outer zone to the inner zone varies with time during the early history, and is still undergoing change long after the pressure at the outer boundary has begun to drop. Curves shown in Fig. 4 do indicate that fluid from the low permeability zone can contribute to production.

Fig. 5 is the build-up curve which would be obtained if the well was shut in at the sand face after 154.8 days of production. Again, as with the drawdown curve, the early portion can be used to calculate kh. The latter portion becomes steeper, indicating the presence of the low permeability rim and the curve terminates with the expected completely flat portion at the shut-in static pressure level for the reservoir.

Fig. 6 is a plot of the rate of change of bottom-hole pressure with cumulative production as a function of cumulative production on log-log graph paper. This type of plot was suggested by L. G. Jones[9] as a means of evaluating reservoir reserves. At a cumulative production of about 17 STB (equivalent to about two days flowing time), the curve becomes nearly flat at a value of 0.74 psi/STB, indicating an initial oil content of about 0.137×10^6 STB, approximately that contained in the inner zone (0.116×10^6 STB). However, the curve then descends again and the final flat portion of the curve occurs at about 0.22 psi/STB, indicating an initial oil content of 0.463×10^6 STB which agrees exactly with that contained in the entire reservoir. The final flat portion of the curve is not achieved until cumulative production has reached about 4,600 STB or cumulative time is about 557 days.

Thus, the short term (less than 80 days) pressure behavior may be generally said to indicate a reservoir of the size of the inner, high permeability zone, while the pressure behavior over a much longer period (of several hundred days duration) is required to estimate reserves in the entire reservoir.

CONCLUSIONS

The analytical solution has been obtained to the problem of flow of a slightly compressible fluid in a limited, composite reservoir with radial symmetry produced by a well at the center. Numerical results have been given for a specific case. These results indicate that for this system (1) fluid can be produced from the low permeability zone, (2) short duration drawdown and build-up data can be analyzed to determine flow capacity and fluid content in the high permeability zone, (3) kh in the low permeability zone cannot be estimated from the conventional semi-log plot, and (4) pressure data over a long period (on the order of several hundred days) is required to estimate fluid content for the entire reservoir including the low permeability zone.

NOMENCLATURE

r_a = radius, cm

$\Delta p_1(r, t_D)$ = pressure drop in inner zone corresponding to dimensionless problem conditions

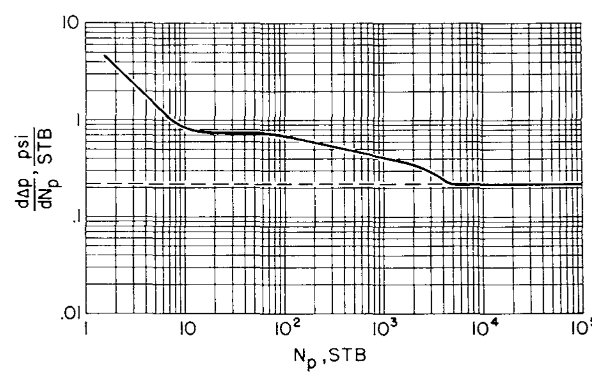

FIG. 6 — DETERMINATION OF RESERVOIR RESERVES BY L. G. JONES' METHOD.

$\Delta p_2(r, t_D)$ = pressure drop in outer zone corresponding to dimensionless problem conditions

$F_k = k_1/k_2$

r_{1a} = radius of interface between inner and outer zones, cm

r_{2a} = radius of outer boundary of system, cm

t_D = dimensionless time, defined by Eq.5c

B_j, C_j = series coefficients defined by Eqs. 4c and 4d

a_j = jth characteristic value or eigenvalue defined as the jth root of Eq. 4e

k_1 = permeability in inner zone, darcys

k_2 = permeability in outer zone, darcys

q = rate of fluid withdrawal from system, cc/sec

h = reservoir thickness, cm

μ = viscosity of flowing fluid, cp

c = compressibility of flowing fluid, atm^{-1}

t = time, seconds

$\Delta p_{1L}, \Delta p_{2L}$ = Laplace transforms of Δp_1 and Δp_2

A, B, C, D = coefficients appearing in Eqs. A-1 and A-2

S = Laplace transforms argument

$r = r_a/r_{2a}$

$r_1 = r_{1a}/r_{2a}$

$r_2 = r_{2a}/r_{2a}$ (unity)

I_0 = modified Bessel function of the first kind, zero order

I_1 = modified Bessel function of the first kind, first order

K_0 = modified Bessel function of the second kind, zero order

K_1 = modified Bessel function of the second kind, first order

J_0 = Bessel function of the first kind, zero order

J_1 = Bessel function of the first kind, first order

Y_0 = Bessel function of the second kind, zero order

Y_1 = Bessel function of the second kind, first order

γ = Euler's constant

$\delta = e^\gamma$

N_{-1} = coefficient of S^{-1} in the numerator of Eq. A-11

N_0 = coefficient of S^0 in the numerator of Eq. A-11

N_1 = coefficient of S^1 in the numerator of Eq. A-11

D_0 = coefficient of S^0 in the denominator of Eq. A-11

D_1 = coefficient of S^1 in the denominator of Eq. A-11

$\Phi_{mn} = Y_m(\sqrt{F_k}\, r_1 a_j)\, J_n(\sqrt{F_k}\, a_j) - Y_n(\sqrt{F_k}\, a_j)\, J_m(\sqrt{F_k}\, r_1 a_j)$

$\Delta p_{1a}(r,t)$ = pressure drop in the inner zone of a specified physical system, atm

$\Delta p_{2a}(r, t)$ = pressure drop in the outer zone of a specified physical system, atm

$\Delta p_{1_{LA}}, \Delta p_{2_{LA}}$ = linear asymptote portions of Δp_1 and Δp_2

$\Delta p_{1_t}, \Delta p_{2_t}$ = transient portions of Δp_1 and Δp_2

B_o = oil formation volume factor, res bbl/STB

ϕ = porosity, fraction

N_p = cumulative oil production

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APPENDIX

PROCEDURE USED IN OBTAINING THE SOLUTION

The general solutions to the Laplace transforms of Eqs. 1a and 1b are known to be:

$$\Delta p_{1L}(r,S) = \left(\frac{A}{S}\right) K_0\left(r\sqrt{S}\right) + \left(\frac{B}{S}\right) I_0\left(r\sqrt{S}\right) \qquad \text{(A-1)}$$

$$\Delta p_{2L}(r,S) = \left(\frac{C}{S}\right) K_0\left(r\sqrt{SF_k}\right) + \left(\frac{D}{S}\right) I_0\left(r\sqrt{SF_k}\right) \qquad \text{(A-2)}$$

where A, B, C and D are functions of S determined by the boundary conditions. Hurst[4] has shown that, as a consequence of the point sink condition, $A = 1$.

Application of the Laplace transforms of the remaining boundary conditions to Eqs. A-1 and A-2 result in the following set of simultaneous equations in B, C and D:

$$B\left[I_0(r_1\sqrt{S})\right] + C\left[-K_0(r_1\sqrt{SF_k})\right] + D\left[-I_0(r_1\sqrt{SF_k})\right] = \left[-K_0(r_1\sqrt{S})\right] \quad \text{(A-3)}$$

$$B\left[I_1(r_1\sqrt{S})\right] + C\left[\frac{1}{\sqrt{F_k}}K_1(r_1\sqrt{SF_k})\right] + D\left[-\frac{1}{\sqrt{F_k}}I_1(r_1\sqrt{SF_k})\right] = \left[K_1(r_1\sqrt{S})\right] \quad \text{(A-4)}$$

$$C\left[-K_1(\sqrt{SF_k})\right] + D\left[I_1(\sqrt{SF_k})\right] = 0 \quad \text{(A-5)}$$

Solution of Eqs. A-3 through A-5 for the coefficients and substitution of these into Eqs. A-1 and A-2 result in the following expression for Δp_{1L} and Δp_{2L}:

$$\Delta p_{1L} = \frac{1}{S}\left[\frac{K_0(r\sqrt{S})\{I_0(r_1\sqrt{S})\Delta_{22} + I_1(r_1\sqrt{S})\Delta_{11}\}}{\{I_0(r_1\sqrt{S})\Delta_{22} + I_1(r_1\sqrt{S})\Delta_{11}\}}\right] + \frac{1}{S}\left[\frac{I_0(r\sqrt{S})\{-K_0(r_1\sqrt{S})\Delta_{22} + K_1(r_1\sqrt{S})\Delta_{11}\}}{\{I_0(r_1\sqrt{S})\Delta_{22} + I_1(r_1\sqrt{S})\Delta_{11}\}}\right] \quad \text{(A-6)}$$

$$\Delta p_{2L} = \frac{1}{S}\left[\frac{\{K_0(r_1\sqrt{S})I_1(r_1\sqrt{S}) + K_1(r_1\sqrt{S})I_0(r_1\sqrt{S})\}\cdot\{I_1(\sqrt{SF_k})K_0(r\sqrt{SF_k}) + K_1(\sqrt{SF_k})I_0(r\sqrt{SF_k})\}}{\{I_0(r_1\sqrt{S})\Delta_{22} + I_1(r_1\sqrt{S})\Delta_{11}\}}\right] \quad \text{(A-7)}$$

where

$$\Delta_{11} = \left[K_0(r_1\sqrt{SF_k})I_1(\sqrt{SF_k}) + I_0(r_1\sqrt{SF_k})K_1(\sqrt{SF_k})\right] \quad \text{(A-8)}$$

$$\Delta_{22} = \frac{1}{\sqrt{F_k}}\left[K_1(r_1\sqrt{SF_k})I_1(\sqrt{SF_k}) - I_1(r_1\sqrt{SF_k})K_1(\sqrt{SF_k})\right] \quad \text{(A-9)}$$

Because the regions for which solutions are desired are bounded, it is to be expected that the solution will be found in the form of a Fourier-Bessel series. This suggests that there is no branch point at the origin of the complex S-plane. Verification is obtained by expanding Δp_{1L} and Δp_{2L} in terms of the following series approximations which are valid near the origin:

$$I_0(z) \simeq 1 + \frac{z^2}{4} + \frac{z^4}{64}$$

$$I_1(z) \simeq \frac{z}{2} + \frac{z^3}{16} + \frac{z^5}{384}$$

$$K_0(z) \simeq -\ln\frac{\delta z}{2} + z^2\left[\frac{1}{4} - \frac{1}{4}\ln\frac{\delta z}{2}\right] + z^4\left[\frac{3}{64} - \frac{1}{64}\ln\frac{\delta z}{2}\right]$$

$$K_1(z) \simeq \frac{1}{z} + z\left[-\frac{1}{4} + \frac{1}{2}\ln\frac{\delta z}{2}\right] + z^3\left[-\frac{5}{64} + \frac{1}{16}\ln\frac{\delta z}{2}\right] \quad \text{(A-10)}$$

where $\delta = e^{\gamma}$ and γ is Euler's constant.

In this way, Δp_{1_L} and Δp_{2_L} are found to have the form:

$$\Delta p_L = \left(\frac{1}{S}\right)\left(\frac{N_{-1}S^{-1} + N_0S^0 + N_1S + \cdots}{D_0S^0 + D_1S + \cdots}\right) \quad \text{(A-11)}$$

which reveals a pole of order 2 at the origin. The residue of the inversion integrand at this pole is evaluated using the formula:[6]

$$\left[S^2 e^{St_D}\Delta p_L\right]'_{S=0} = \textbf{Residue at the origin} \quad \text{(A-12)}$$

In this case, the residue will have the form:

$$t_D\left(\frac{N_{-1}}{D_0}\right) + \left(\frac{N_0D_0 - N_{-1}D_1}{D_0^2}\right) = \textbf{Residue at the origin} \quad \text{(A-13)}$$

which is the linear time asymptote portion of the solution that will constitute the solution after the transient has disappeared.

Eqs. A-10 contain sufficient numbers of terms that the coefficients in Eq. A-13 may be evaluated completely. When this is done the linear asymptotes for Δp_1 and Δp_2 may be written. They are found to be:

$$\Delta p_{1_{LA}}(r, t_D) = 2t_D + \ln\frac{r_1}{r} + \frac{r^2}{2} - F_k\left[\ln r_1 + \frac{3}{4}\right] + (F_k - 1)\left[r_1^2 - \frac{r_1^4}{4}\right] \quad \text{(A-14)}$$

$$\Delta p_{2_{LA}}(r, t_D) = 2t_D + F_k\left[\frac{r^2}{2} - \ln r - \frac{3}{4}\right] + (F_k - 1)\left[\frac{r_1^2}{2} - \frac{r_1^4}{4}\right] \quad \text{(A-15)}$$

Along the negative real axis, Δp_{1_L} and Δp_{2_L} will have simple poles at the zeros of the denominator (which is the same for Δp_{1_L} and Δp_{2_L}) if each of these terms is to satisfy Eqs. 1a or 1b. The imaginary part of the denominator vanishes along the negative real axis and the zeros of the denominator are found to occur at the zeros of Eq. 4e. The residues of $e^{St_D}\Delta p_{1_L}$ and $e^{St_D}\Delta p_{2_L}$ at these poles were evaluated using the following relation applicable at simple poles: residue of $[f(S)/g(S)] = [f(S_n)/g'(S_n)]$ where the S_n are the zeros of $g(S)$, and $g'(S_n) \neq 0$.

The resulting series of terms constitutes the transient portions of the solutions. These series have the form:

$$\Delta p_{1_T} = \sum_{j=1}^{\infty} B_j e^{-\alpha_j^2 t_D} \quad \text{(A-16)}$$

$$\Delta p_{2_T} = \sum_{j=1}^{\infty} C_j e^{-\alpha_j^2 t_D} \quad \text{(A-17)}$$

where B_j and C_j are defined by Eqs. 4c and 4d.

In this portion of the work, the complex relationships between Bessel functions and modified Bessel functions were used.[2,6]

Considerable algebraic manipulation was involved in arriving at the compact form given for B_j and C_j. This procedure was facilitated by the use of the following relations which are easily derived from the commonly known properties of Bessel functions:[2,6]

$$\frac{d\Phi_{m,n}}{d\alpha_j} = \sqrt{F_k}\left[r_1\Phi_{m-1,n} + \Phi_{m,n-1}\right] - \left(\frac{m+n}{\alpha_j}\right)\Phi_{m,n}$$

$$\frac{d\Phi_{m,n}}{d\alpha_j} = -\sqrt{F_k}\left[r_1\Phi_{m+1,n} + \Phi_{m,n+1}\right] + \left(\frac{m+n}{\alpha_j}\right)\Phi_{m,n}$$

$$\frac{d\Phi_{m,n}}{d\alpha_j} = \sqrt{F_k}\left[\Phi_{m,n-1} - r_1\Phi_{m+1,n}\right] + \left(\frac{m-n}{\alpha_j}\right)\Phi_{m,n}$$

where $\Phi_{m,n}$ is a combination of Bessel functions defined in the Nomenclature. Also, repeated use was made of Eq. 4e in arriving at the final forms B_j and C_j.

The complete solutions (Eqs. 4a and 4b) are the sums:

$$\Delta p_1(r, t_D) = \Delta p_{1_{LA}} + \Delta p_{1_T} \quad \text{(A-18)}$$

$$\Delta p_2(r, t_D) = \Delta p_{2_{LA}} + \Delta p_{2_T} \quad \text{(A-19)}$$

It can be shown that the final solutions satisfy the partial differential equations and the boundary conditions. In test problems, the initial condition was always found to be properly satisfied. As a further verification $\lim_{r_1 \to 1} \Delta p_1$ and $\lim_{F_k \to 1} \Delta p_1, \Delta p_2$ were considered. In both instances, the resulting solution is identical with the corresponding single-zone solution given by Muskat.[7] Finally, it can be shown that Eq. A-14 is equivalent to the solution obtained to the same problem by Hopkinson *et al.*[8] Ref. 8, however, contains only the linear asymptote solution of the inner zone.

APPROXIMATIONS FOR $\Delta p_1(r, t_D)$ FOR SMALL r AND SMALL t_D

In this r–t_D region, the series convergence is slow. It is readily shown from $\Delta p_{1L}(r, s)$ that, for t_D small:

$$\Delta p_1(r, t_D) \simeq \frac{1}{2}\left[-Ei\left(-\frac{r^2}{4t_D}\right)\right] \quad \text{(A-20)}$$

PROCEDURE FOR NUMERICAL EVALUATION OF THE SOLUTION FOR A SPECIFIC CASE

1. The left-hand side of Eq. 4e was evaluated at a series of values of a to obtain estimates of the values of a_j.
2. More exact values of a_j were obtained by a modified Newton method process, which is characterized by the following equations:

$$f(\alpha) + (\delta\alpha)\left[\frac{f(\alpha_1) - f(\alpha_2)}{\alpha_1 - \alpha_2}\right] = 0$$

$$\alpha_j^1 = \alpha_j^0 + (\delta\alpha_j)^0$$

The interval $(a_1 - a_2)$ was chosen to be a very small fraction of the oscillatory period of the left-hand side of Eq. 4e. The value a_j was assumed to have been computed when (δa) became suitably small.

3. $\Delta p_1(r, t_D)$ and $\Delta p_2(r, t_D)$ were computed for a series of values of r and t_d.

Present use of the solution does not justify making the entire process an automatic one contained in a single program, although this could be done. Bessel function subroutines prepared by General Motors Corp. were used in the machine computations.

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